Integrieren, Integrationstechniken

Ist $F'(x) = f(x)$ für alle x aus einem Intervall $I = [a,b]$, so heißt $F(x)$ eine Stammfunktion von f.

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \left(= \left[F(x)\right]_a^b \right)$$

Ist $F(x)$ eine Stammfkt von $f(x)$ so ist $G(x) = F(x) + c$ (c konstante) ebenfalls eine Stammfkt.

- Rechenregeln

$$\int_a^b c \cdot f(x)\,dx = c \cdot \int_a^b f(x)\,dx$$

$$\int_a^b \left(f(x) + g(x) \right) dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b f(cx)\,dx = \frac{1}{c}\left(F(c \cdot b) - F(c \cdot a) \right) = \left[\frac{1}{c} F(cx)\right]_a^b$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

für alle $a, b, c \in I$

$$\left(\int_a^x f(t)\,dt \right)' = f(x) \quad \text{und}$$

$$\left(\int_x^b f(t)\,dt \right)' = -f(x)$$

Bemerkung Die Integrationsvariable x in $\int_a^b f(x)\,dx$ kann durch jeden anderen Buchstaben ersetzt werden. Es ist eine "dummy" Variable, vgl. Summationsindex in $\sum_{i=0}^{n} f(i) = \sum_{k=0}^{n} f(k) = \sum_{j=0}^{n} f(j)$

Bsp 1) $\int_0^2 (x^4 - 5x^3 - 1)\,dx =$

$f(x) = 1$
$F(x) = x$
da $x' = 1$

$$\int_0^2 x^4\,dx - 5\int_0^2 x^3 - \int_0^2 1\cdot dx$$

$$= \left[\frac{x^5}{5}\right]_0^2 - 5\left[\frac{x^4}{4}\right]_0^2 - [x]_0^2$$

$$= \left(\frac{2^5}{5} - \frac{0^5}{5}\right) - 5\left(\frac{2^4}{4} - \frac{0^4}{4}\right) - (2-0)$$

$$= \frac{32}{5} - 5\cdot\cancel{\frac{16}{4}}^{4} - 2 = \frac{32}{5} - 5\cdot 4 - 2 =$$

$$= \frac{32}{5} - {}^{5}22 = \frac{32-110}{5} = -\frac{78}{5}$$

2) $\int_{-1}^{1} (x - e^{-x})\,dx = \int_{-1}^{1} x\,dx - \int_{-1}^{1} e^{-x}\,dx$

$$= \left[\frac{x^2}{2}\right]_{-1}^{1} - \left[-e^{-x}\right]_{-1}^{1}\ (*)$$

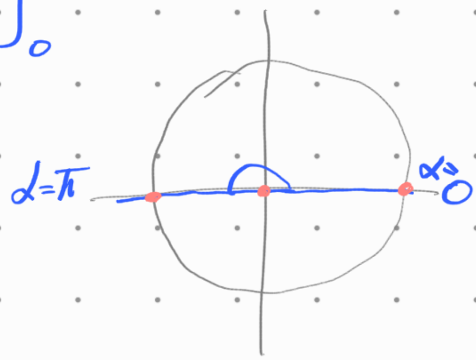
(*) Probe: $(-e^{-x})' = -(e^{-x})(-x)' = -(-e^{-x}) = e^{-x}$

$$= \left(\frac{1^2}{2} - \frac{(-1)^2}{2}\right) - \left(-e^{-1} - \left(-e^{-(-1)}\right)\right)$$

$$= \left(\cancel{\tfrac{1}{2}-\tfrac{1}{2}}\right) - \left(-\tfrac{1}{e}+e\right) = \tfrac{1}{e} - e$$

3) $$\int_0^1 \cos(\underbrace{\pi}_{c=\pi} x)\,dx = \frac{1}{\pi}\left[\sin(\pi x)\right]_0^1$$

$$= \frac{1}{\pi}\left(\underbrace{\sin \pi}_{=0} - \underbrace{\sin 0}_{0}\right) = \frac{1}{\pi}\cdot 0 = 0$$



▶ Unbestimmtes Integral

$$\int f(x)\,dx = F(x) + c$$ ← eine Konstante

die "Menge" aller Stammfunktionen von $f(x)$

Bsp 1) $$\int x^5\,dx = \frac{x^6}{6} + c$$

2) $$\int \cos^2 x\,dx = *$$

$$\boxed{\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ \Rightarrow \cos^2 x &= \frac{1+\cos 2x}{2} \end{aligned}}$$

? Additionstheorem

$\cos(x+y) = \cos x \cos y - \sin x \cdot \sin y$

falls $x=y$:

$\cos 2x = \cos^2 x - \sin^2 x$

Trig. Pythagoras $\sin^2 x + \cos^2 x = 1$

$$* = \int \frac{1+\cos 2x}{2}\,dx = \frac{1}{2}\int 1\,dx + \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{1}{2}x + \frac{1}{2}\cdot\frac{1}{2}\sin 2x + C$$

Substitutionsregel

$$\int_a^b f(g(x))\cdot \underbrace{g'(x)\,dx}_{dg} = \Big[F(g(x))\Big]_a^b$$

$$= F(g(b)) - F(g(a))$$

$$\boxed{df = f'(x)\,dx}$$

Probe: $\big(F(g(x))\big)' \overset{\text{Kettenregel}}{=} f(g(x))\cdot g'(x)$

$$\int_{g(a)}^{g(b)} f(g)\,dg = \Big[F(g)\Big]_{g(a)}^{g(b)} = \Big[F(g(x))\Big]_a^b$$

Bsp 1) $\displaystyle\int_0^{\pi/2} \cos x \cdot e^{\sin x}\,dx = \left[\begin{array}{l} f(x) = e^x \to F(x) = e^x \\ g(x) = \sin x \\ g'(x) = \cos x \end{array}\right]$

ich kann erkennen

$$= \Big[e^{\sin x}\Big]_0^{\pi/2} = \left(e^{\underbrace{\sin\frac{\pi}{2}}_{1}} - e^{\underbrace{\sin 0}_{0}}\right) = e^1 - e^0 = e - 1$$

2) $\displaystyle\int_0^3 \sqrt{x+1}\,dx = \int_0^3 (x+1)^{\frac{1}{2}}\,dx$

$$= \left[\begin{array}{l} g = x+1 \\ dg = g'\,dx = 1\cdot dx = dx \end{array}\middle|\begin{array}{l} x=0 \to g=1 \\ x=3 \to g=4 \end{array}\right]$$

$$= \int_1^4 g^{\frac{1}{2}}\,dg = \left[\frac{g^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^4 = \left[\frac{g^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_1^4$$

$$= \frac{2}{3}\left(4^{\frac{3}{2}} - \underbrace{1^{3/2}}_{=1}\right) = \frac{2}{3}\left(\sqrt{4^3} - 1\right) = \frac{2}{3}(8-1) = \frac{14}{3}$$

3) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{3}} \tan x\,dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin x}{\cos x}\,dx$

Minus verwende

$$= \left[\begin{array}{l|l} \colorbox{pink}{$u = \cos x$} & x = -\frac{\pi}{4} \to u = \frac{\sqrt{2}}{2} \\ du = u' dx & \\ = (-\sin x)\, dx & x = \frac{\pi}{3} \to u = \frac{1}{2} \\ \colorbox{yellow}{$\sin x \cdot dx = -du$} & \end{array}\right] = \int_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}} \frac{-du}{u}$$

um die Grenzen zu tauschen

$$= \int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \frac{1}{u}\, du = \Big[\ln|u|\Big]_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} =$$

$$= \left(\ln\frac{\sqrt{2}}{2} - \ln\frac{1}{2}\right) = \ln\sqrt{2} - \ln 2 - (\underbrace{\ln 1}_{=0} - \ln 2)$$

$$= \ln\sqrt{2} = \ln 2^{\frac{1}{2}} = \frac{1}{2}\ln 2$$

4) $$\int_2^4 (\underbrace{2x-5}_{=u})^5\, dx = \left[\begin{array}{l|l} u = 2x-5 & x = 2 \to u = -1 \\ du = u' dx = 2dx & x = 4 \to u = 3 \\ dx = \frac{du}{2} & \end{array}\right]$$

$$= \int_{-1}^{3} u^5 \frac{du}{2} = \frac{1}{2}\left[\frac{u^6}{6}\right]_{-1}^{3} = \frac{1}{2}\left(\frac{3^6}{6} - \frac{(-1)^6}{6}\right)$$

$$= \frac{729 - 1}{12} = \frac{728}{12}$$

▶ Partielle Integration

$$\int_a^b f(x)\, \underbrace{dg(x)}_{g'(x)\,dx} = \Big[f(x)\cdot g(x)\Big]_a^b - \int_a^b g(x)\, \underbrace{df(x)}_{f'(x)\,dx}$$

$$\int_a^b f\, dg = [fg]_a^b - \int_a^b g\, df$$

Bsp 1) $$\int_1^2 x^3 \cdot \ln x\, dx = \left[\begin{array}{l} f = x^3 \to df = f' dx = 3x^2 dx \\ dg = \ln x\, dx \to g = ??? \end{array}\right]$$

Suche g mit $g' = \ln x$

↯

nochmal $$\int_1^2 x^3 \ln x\, dx = \left[\begin{array}{l} f = \ln x \to df = \frac{1}{x} dx \\ \underbrace{dg = x^3 dx}_{g' = x^3} \to g = \frac{x^4}{4} \end{array}\right]$$

$$= \left[(\ln x)\cdot \frac{x^4}{4}\right]_1^2 - \int_1^2 \frac{x^4}{4}\cdot\frac{1}{x}\,dx$$

$$= \left[\frac{x^4}{4}\ln x\right]_1^2 - \frac{1}{4}\int_1^2 x^3\,dx = \left[\frac{x^4}{4}\ln x\right]_1^2 - \frac{1}{4}\left[\frac{x^4}{4}\right]_1^2$$

$$= \left(\frac{16}{4}\ln 2 - \underbrace{0}_{\ln 1 = 0}\right) - \frac{1}{4}\left(\frac{16}{4} - \frac{1}{4}\right) = 4\ln 2 - \frac{15}{16}$$

2) $$\int_0^1 x e^x\,dx = \begin{bmatrix} f = x \rightarrow df = dx \\ dg = e^x dx \rightarrow g = e^x \end{bmatrix}$$

$$= \left[x e^x\right]_0^1 - \int_0^1 e^x\,dx = \left[x e^x\right]_0^1 - \left[e^x\right]_0^1$$

$$= \left(1\cdot e^1 - 0\right) - \left(e^1 - e^0\right) = e - e + 1 = 1$$

3) $$\int_0^\pi x\cdot \sin x\,dx = \begin{bmatrix} f = x \rightarrow df = dx \\ dg = \sin x\,dx \rightarrow g = -\cos x \end{bmatrix}$$

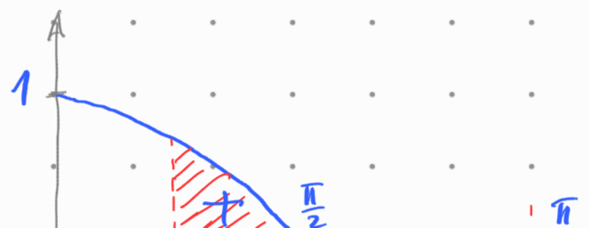
$$= \left[-x\cdot\cos x\right]_0^\pi - \int_0^\pi (-\cos x)\cdot dx$$

$$= \left[-x\cdot\cos x\right]_0^\pi + \left[\sin x\right]_0^\pi = \left(-\pi\cdot\cos\pi - 0\right) + \left(\underbrace{\sin\pi}_{=0} - \underbrace{\sin 0}_{=0}\right)$$

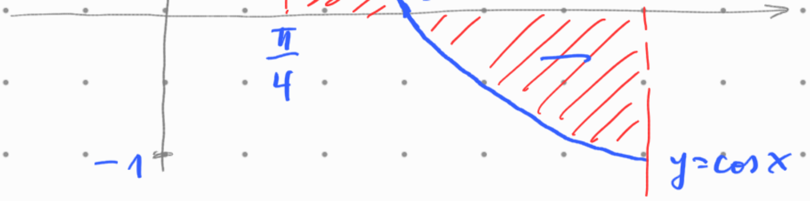
$$= -\pi(-1) = \pi$$

Fläche unter dem Graphen

Bsp



Bestimme die Fläche zwischen den Graphen von $y = \cos x$ und



der x-Achse für $\frac{\pi}{4} \leq x \leq \pi$.

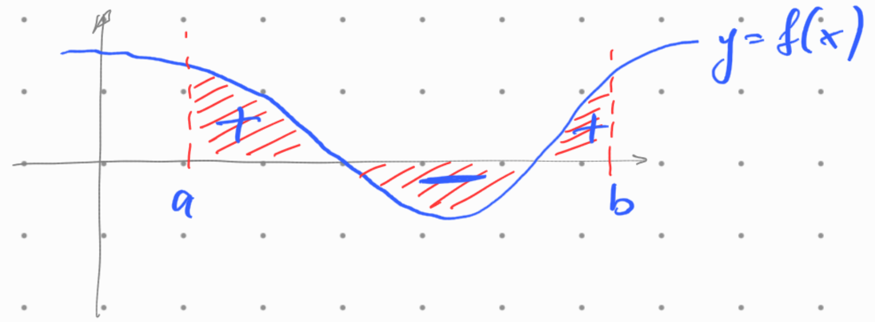
1) $\int_{\frac{\pi}{4}}^{\pi} \cos x\, dx = [\sin x]_{\frac{\pi}{4}}^{\pi} = \sin\pi - \sin\frac{\pi}{4} = 0 - \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}$ ↯

2) $F_1 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\, dx = [\sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin\frac{\pi}{4} = 1 - \frac{\sqrt{2}}{2}$

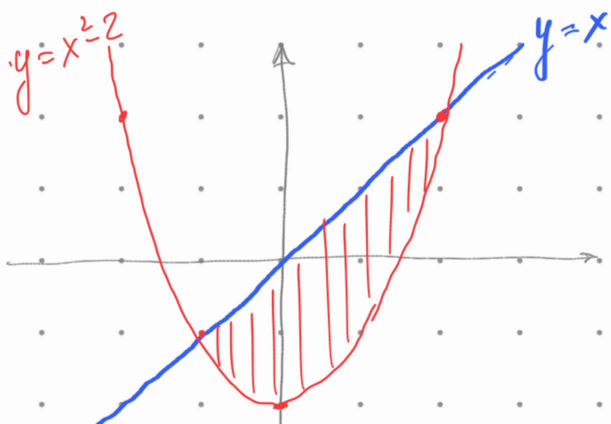
$F_2 = -\int_{\frac{\pi}{2}}^{\pi} \cos x\, dx = -[\sin x]_{\frac{\pi}{2}}^{\pi} = -\left(\sin\pi - \sin\frac{\pi}{2}\right) = -(0-1) = 1$

$\Rightarrow \quad F_1 + F_2 = 1 - \frac{\sqrt{2}}{2} + 1 = 2 - \frac{\sqrt{2}}{2}$

i. A. ist $\int_a^b f(x)$ gleich dem Flächeninhalt zwischen dem Graphen von f, x-Achse und den Geraden $x=a$ und $x=b$, wobei die Teile unterhalb der x-Achse negativ gezählt sind.



Bsp Berechne die Fläche zwischen der Parabel $y = x^2 - 2$ und der Geraden $y = x$



1) Schnittpunkte

$x^2 - 2 = x$

$x^2 - x - 2 = 0 \Rightarrow x_1 = -1, x_2 = 2$

2) Da der Graph von $f(x) = x$

oberhalb des Graphen von $g(x)=x^2-2$ liegt, ist $f(x)-g(x)\geq 0$ für $-1 \leq x \leq 2$.

3) Die Fläche ist

$$F=\int_{-1}^{2}\left(f(x)-g(x)\right)dx=\int_{-1}^{2}\left(x-x^2+2\right)dx=$$

$$=\left[\frac{x^2}{2}\right]_{-1}^{2}-\left[\frac{x^3}{3}\right]_{-1}^{2}+2\left[x\right]_{-1}^{2}=\left(\frac{4}{2}-\frac{1}{2}\right)-\left(\frac{8}{3}-\frac{-1}{3}\right)+2\left(2-(-1)\right)$$

$$=\frac{3}{2}-\frac{9}{3}+6=6-3+\frac{3}{2}=3+\frac{3}{2}=\frac{9}{2}$$